

Bose–Einstein Condensation and Thermalization of the Quark Gluon Plasma

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Abstract

In ultra-relativistic heavy ion collisions, the matter formed shortly after the collision is a dense, out of equilibrium, system of gluons characterized by a semi-hard momentum scale Q_s . Simple power counting arguments indicate that this system is over-occupied: the gluon occupation number is parametrically large when compared to a system in thermal equilibrium with the same energy density. On short time scales, soft elastic scatterings tend to drive the system towards the formation of a Bose–Einstein condensate that contains a large fraction of the gluons while contributing little to the energy density. The lifetime and existence of this condensate depends on whether inelastic processes, that occur on the same time scale as the elastic ones, preferably increase or decrease the number of gluons. During this overpopulated stage, and all the way to thermalization, the system behaves as a strongly interacting fluid, even though the elementary coupling constant is small. We argue that while complete isotropization may never be reached, the system may yet evolve for a long time with a fixed anisotropy between average longitudinal and transverse momenta.

1 Introduction

One of the central theoretical issues in the description of heavy ion collisions is to understand how the partons that are freed by the collisions evolve into a thermalized system amenable to an hydrodynamical description. Let us recall that most of the produced partons originate from the small x components of the wavefunctions, that are dominated by gluon saturation and occupation numbers of order $1/\alpha_s$ [1–3]. Such wavefunctions are well described by the Color Glass Condensate (CGC) effective field theory [4]. This effective theory allows in particular for the calculation of the energy-momentum tensor immediately after the collision. Because the chromo-electric and

chromo-magnetic fields immediately after the collision are collinear to the collision axis, a configuration called “Glasma” [5], the energy-momentum tensor is of the form $T^{\mu\nu} = \text{diag}(\epsilon, \epsilon, \epsilon, -\epsilon)$, and therefore has a negative longitudinal pressure at very early times. Such an anisotropy between the transverse and longitudinal pressures precludes a direct use of this energy-momentum tensor as initial condition for hydrodynamics, because the matching between the Glasma and the hydrodynamical evolution would require viscous corrections that are as large as the ideal terms.

However this particular form of the energy momentum tensor, and the underlying structure of the fields, is not expected to last for a period of time much longer than $1/Q_s$, where Q_s is the saturation scale. In fact, instabilities of various kinds [6] may lead rapidly (over the same time scale $1/Q_s$) towards an isotropic energy momentum tensor. But even if isotropization of the energy-momentum tensor indeed occurs one outstanding issue remains, namely whether the phase space distribution functions relaxes towards the equilibrium Bose-Einstein distributions or not. The “bottom-up” scenario [7] provides a systematic way by which this relaxation occurs as a result of hard elastic and inelastic collisions. In ref. [8], it was demonstrated that anisotropy driven instabilities can significantly alter this picture. In this paper, we would like to address this question from a different point of view.

Our paper is motivated by the basic observation that initially the gluon density in the Glasma is parametrically large compared to the value it should have in a system in thermal equilibrium with the same energy density. In systems where collisions conserve the particle number, any over (or under) -population in the initial condition can be accommodated by the appearance of a chemical potential in the equilibrium distribution. However, as we shall show, the initial over-population in the Glasma is so large, when the coupling constant is small, that the maximum allowed value for the chemical potential, that is $\mu = m$ (where m is a medium generated mass for gluons) is insufficient to account for the excess of gluons. This tension may be resolved in part by the dynamical generation of a Bose-Einstein condensate, corresponding to a large occupation of the zero momentum mode, and in part by inelastic processes that in the long run tend to tame the particle excess.

After a short time scale of order $1/Q_s$, at which instabilities in the Glasma are expected to isotropize the system, it should be possible to describe the Glasma with color singlet distribution functions for both the particle content and the condensate. We will consider a generic form for such a distribution that has the following features: (i) it is dilute above some hard scale Λ , (ii) the gluon occupation number is saturated at a value $1/\alpha_s$ below some “coherence scale” Λ_s , and (iii) between these two scales the gluon distribution is inversely proportional to the energy. At early times, $\Lambda_s \sim \Lambda \sim Q_s$. In the expanding system, both Λ and Λ_s decrease in time; Λ_s decreases faster than Λ , so that thermalization is achieved by depletion of high energy modes. We will estimate the time it takes for Λ_s to become of order $\alpha_s \Lambda$, and will argue further that this coincides with the thermalization time: in this picture, thermalization corresponds to the situation where the distribution function has acquired the equilibrium shape for most of the relevant momenta, thereby maximizing the entropy. We will argue that during the evolution to thermalization, because of the high occupancy of the low momentum modes, the system remains strongly interacting, although the coupling constant is small. It may also develop a fixed anisotropy ratio of $\langle p_z^2 \rangle$ relative to $\langle p_T^2 \rangle$.

2 The overpopulated quark-gluon plasma

In the CGC description of heavy ion collisions, the gluons that contribute dominantly to the energy density are freed over a time scale of order $\tau_0 \sim Q_s^{-1}$, with Q_s the saturation scale [9]. These gluons have typical transverse momenta of order Q_s , and an energy density

$$\epsilon_0 = \epsilon(\tau = Q_s^{-1}) \sim \frac{Q_s^4}{\alpha_s}. \quad (1)$$

The number of gluons produced per unit volume is given by

$$n_0 = n(\tau = Q_s^{-1}) \sim \frac{Q_s^3}{\alpha_s}, \quad (2)$$

so that the average energy per gluon is indeed $\epsilon_0/n_0 \sim Q_s$. In fact Q_s is initially the only scale characterizing the system, and the initial phase distribution function, a dimensionless object, is of the form $f_0(\mathbf{p}/Q_s, \mathbf{x}Q_s, tQ_s \sim 1)$.

One may characterize the initial distribution of gluons by the dimensionless combination $n_0 \epsilon_0^{-3/4} \sim 1/\alpha_s^{1/4}$. In comparison, in an equilibrated system of gluons at temperature T ,

$$\epsilon_{\text{eq}} \sim T^4, \quad n_{\text{eq}} \sim T^3. \quad (3)$$

and

$$n_{\text{eq}} \epsilon_{\text{eq}}^{-3/4} \sim 1. \quad (4)$$

There is therefore a mismatch, by a large factor $\alpha_s^{-1/4} \gg 1$ (in weak coupling asymptotics, $\alpha_s \ll 1$), between the value of $n\epsilon^{-3/4}$ in the initial condition and that in an equilibrated system of gluons. We interpret this mismatch as an “overpopulation” of the initial distribution. Since the gluons are bosons, it is natural to explore the possibility that the system copes with this overpopulation with a new equilibrium state involving a Bose condensate.

At this time, we assume that the equilibrium state is reached by elastic processes that conserve the number of gluons, and consequently introduce a chemical potential in the equilibrium distribution function that one is looking for. The energy density and number density read then

$$\epsilon_{\text{eq}} = \int_{\mathbf{p}} \omega_{\mathbf{p}} f_{\text{eq}}(\mathbf{p}) \quad , \quad n_{\text{eq}} = \int_{\mathbf{p}} f_{\text{eq}}(\mathbf{p}). \quad (5)$$

where

$$f_{\text{eq}}(\mathbf{k}) \equiv \frac{1}{e^{\beta(\omega_{\mathbf{k}} - \mu)} - 1}. \quad (6)$$

The temperature $T = 1/\beta$ and the chemical potential are adjusted so as to reproduce the initial values of ϵ_0 and n_0 . Here $\omega_{\mathbf{p}}$ is the energy of a gluon with momentum \mathbf{p} . An important feature of any dense system of gluons is that, as a result of their many-body interactions, gluons develop effective medium dependent masses, that is, $\omega_{\mathbf{p}=0} = m \neq 0$. For instance, in a weakly interacting system of gluons in thermal equilibrium, this mass can be obtained in the well known Hard Thermal Loop (HTL) approximation [10], where $m \sim \alpha_s^{1/2} T$. For the initial distribution, we estimate the gluon mass as

$$m_0^2 \sim \alpha_s \int_{\mathbf{p}} \frac{df_0}{d\omega_p} \sim Q_s^2, \quad (7)$$

whereas, when the plasma has reached equilibrium, $m \sim \alpha_s^{1/2} T \sim \alpha_s^{1/4} Q_s$, with $T \sim \epsilon_{\text{eq}}^{1/4} \sim Q_s/\alpha_s^{1/4}$. Note that in this discussion we shall make no distinction between the mass defined from the spectrum, $m = \omega_{\mathbf{p}=0}$, and the screening mass. Both are parametrically comparable, and we shall often refer to m as the “Debye mass”.

Since $f_{\text{eq}}(\mathbf{k})$ is a growing function of the chemical potential, one way to cope with an excess of particles is to have a positive chemical potential. Note however that the chemical potential cannot grow larger than m , or else f_{eq} would become negative. There is therefore a maximum number density that can be accommodated by the introduction of a chemical potential at a given T ,

$$n_{\text{max}} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{e^{\beta(\omega_{\mathbf{k}} - m)} - 1} \sim T^3. \quad (8)$$

In this last estimate, we have used that $m \ll T$, so that the integrals are dominated by the region $m \lesssim p \lesssim T$ where $T/(\omega_{\mathbf{p}} - m) \sim T/p$. This calculation reveals that $n_{\text{max}} \sim Q_s^3/\alpha_s^{3/4}$ remains parametrically smaller than the initial gluon density $n_0 \sim Q_s^3/\alpha_s$.

We are then led to the conclusion that, when the gluons undergo only elastic collisions, Bose condensation occurs, with the equilibrium distribution function taking the form

$$f_{\text{eq}}(\mathbf{k}) = n_c \delta(\mathbf{k}) + \frac{1}{e^{\beta(\omega_{\mathbf{k}} - m_0)} - 1}, \quad (9)$$

with n_c the density of particles in the condensate, defined as the difference between the total density and the density of “thermal particles” (i.e., n_{max}). In fact, most of the particles are to be found in the condensate. We have indeed

$$n_c \sim \frac{Q_s^3}{\alpha_s} \left(1 - \alpha_s^{1/4}\right). \quad (10)$$

Note however that the condensate particles carry only a small fraction of the total energy density, since the energy of the condensed particles is

$$n_c m \sim \frac{Q_s^3}{\alpha_s} \alpha_s^{1/4} Q_s \sim \alpha_s^{1/4} T^4 \ll \epsilon_0. \quad (11)$$

The formation of a condensate is intimately associated with particle number conservation. When inelastic processes occur sufficiently rapidly, the number of gluons is not conserved anymore. Under such circumstance, no chemical potential can appear in the distribution function, and neither can a singular component representing the Bose condensate develop. The only equilibrium function is of the form of Eq. (6) with $\mu = 0$. In order to reach this equilibrium distribution, the system has to decrease its number of particles, via inelastic processes.

It is also instructive to look at the entropy of the system, $s \sim \int_{\mathbf{p}} \ln f_{\mathbf{p}}$. This is dominated by the hard momenta, so that initially, $s \sim Q_s^3$ (to within logarithmic corrections). In the case where there is no condensate, the equilibrium entropy is simply $s \sim T^3 \sim Q_s^3/\alpha_s^{3/4}$. As expected, there is entropy increase as the shape of the momentum distribution evolves from the initial distribution towards the thermal distribution. Note that this increase is accompanied by the decrease of the particle number, so that in equilibrium $s \sim T^3 \sim n$. When elastic collisions dominate, a condensate forms. The condensate carries no entropy, and absorbs the excess particles. The equilibrium state is characterized by the same equilibrium entropy $s \sim T^3$, but now this entropy is carried only by the

thermal particles, that is, $s \sim n_{\text{max}}$. In both cases, when equilibrium is reached the overpopulation disappears, as it should, namely, $n_g \epsilon^{-3/4} \sim 1$, with $n_g \sim n$ when there is no condensate and $n_g \sim n_{\text{max}}$ in the presence of a condensate.

The thermodynamical considerations of the present section lead us to expect two possible equilibrium states, given the initial condition. Either a system with a Bose condensate, if the approach to equilibrium is driven by elastic collisions, or a system with fewer number of particles if inelastic processes are important. Note however that the presence of inelastic, particle number changing, processes does not preclude the possibility that a transient condensate develops during the evolution of the system. This is a dynamical issue that depends on the respective rates of particle production versus particle annihilation processes. We explore this question in the next sections.

3 Kinetic evolution dominated by elastic collisions

In order to address the question of how the system evolves towards its equilibrium state, we shall rely on a simple kinetic description based on the following transport equation

$$\partial_t f(\mathbf{k}, X) = C_{\mathbf{k}}[f] , \quad (12)$$

where $C_{\mathbf{k}}[f]$ is the usual collision integral. We ignore at this point all drift terms in the left hand side. We assume that initially the system is isotropic, a property that is preserved by the evolution. In this section we focus on a gluon system in a non-expanding box. The effect of longitudinal expansion will be discussed later. We also assume that gluons undergo only elastic collisions, deferring the discussion of the effect of inelastic number changing processes to the next section.

Our main goal is to understand how collisions drive the initial distribution towards local equilibrium, and get a measure of the basic time scales involved. A detailed answer can only be obtained through explicit numerical solutions of the Boltzmann equation. Results of such calculations will indeed be presented elsewhere [11]. Here, we shall argue that we can capture the dominant qualitative features of the solution by assuming that the evolution is dominated by only two scales, Λ_s and Λ . We shall assume that the $1/\omega_{\mathbf{p}}$ thermal distribution gradually builds up from energy Λ_s where the distribution function is $\sim 1/\alpha_s$ down to Λ . To be concrete, although such an explicit form is not really needed in our arguments, we may assume that at all times $t > 1/Q_s$, the distribution function takes the form

$$f(p) \sim \frac{1}{\alpha_s} \text{ for } p < \Lambda_s, \quad f(p) \sim \frac{1}{\alpha_s} \frac{\Lambda_s}{\omega_{\mathbf{p}}} \text{ for } \Lambda_s < p < \Lambda, \quad f(p) \sim 0 \text{ for } \Lambda < p. \quad (13)$$

At $t \sim 1/Q_s$, both scales Λ_s and Λ coincide with Q_s . As time progresses, the two scale separates, with Λ_s decreasing quickly, and Λ evolving much more slowly. Thermalization is reached when $\Lambda_s/\Lambda \sim \alpha_s$, at which point, $f(\Lambda)$ becomes of order unity.

A more precise definition of these two scales can be obtained by looking more closely at the collision integral. In the small angle approximation, assuming $2 \rightarrow 2$ elastic scattering and isotropy, standard manipulations lead to [11]

$$\left. \frac{\partial f}{\partial t} \right|_{\text{coll}} \sim \frac{\Lambda_s^2 \Lambda}{p^2} \partial_p \left\{ p^2 \left[\frac{df}{dp} + \frac{\alpha_s}{\Lambda_s} f(p)(1 + f(p)) \right] \right\}. \quad (14)$$

The fixed point solution of this equation is a Bose-Einstein distribution with temperature $T = \Lambda_s/\alpha_s$ (and indeed at thermalization, $T \sim \Lambda \sim \Lambda_s/\alpha_s$). The two scales Λ_s and Λ may be obtained from the integrals

$$\frac{\Lambda\Lambda_s}{\alpha_s} \equiv - \int_0^\infty dp p^2 \frac{df}{dp}, \quad \frac{\Lambda\Lambda_s^2}{\alpha_s^2} \equiv \int_0^\infty dp p^2 f(1+f). \quad (15)$$

Remarkably, in the regime where $f \gg 1$ ($f \sim 1/\alpha_s$), all dependence on α_s drops from the collision integral.

By taking moments of the collision integral above with arbitrary powers of p , it is not difficult to show that the typical collision time is given by

$$t_{\text{scat}} = \frac{\Lambda}{\Lambda_s^2}, \quad (16)$$

which is independent of α_s . This collision time should not be confused with the thermalization time that we shall define later, and which depends on α_s . The collision time t_{scat} is also a function of time that we shall determine shortly.

In all parametric estimates to be done below, we shall exploit the fact that the integrals are dominated by the largest momenta, of order Λ , large compared to the Debye mass, so that the distribution in the interesting region can just be taken to be $f(p) \sim \Lambda_s/(\alpha_s p)$ up to a cut-off of order Λ . Thus the number of gluons associated with this distribution is simply

$$n_g \sim \frac{1}{\alpha_s} \Lambda^2 \Lambda_s. \quad (17)$$

In addition, one has the contribution of the condensate, n_c ,

$$n = n_c + n_g, \quad (18)$$

with initially $n = n_0$ and $n_c = 0$. Similarly, the energy density in gluon modes is

$$\epsilon_g \sim \frac{1}{\alpha_s} \Lambda_s \Lambda^3, \quad (19)$$

to which should be added the energy of the condensate

$$\epsilon_c \sim m n_c \quad (20)$$

in order to get the full energy density. To determine ϵ_c we need to estimate m ,

$$m^2 \sim \alpha_s \int dp p^2 \frac{df(p)}{d\omega_p} \sim \Lambda \Lambda_s, \quad (21)$$

so that

$$\epsilon_c \sim n_c \sqrt{\Lambda_s \Lambda}. \quad (22)$$

Initially, when n_c is small, ϵ_c represents a small correction to the energy density carried by the gluons. As time goes on, n_c increases but ϵ_c remains small, as we shall verify.

Our determination of the time dependence of the scales Λ_s and Λ will rely on energy conservation,

$$\Lambda_s \Lambda^3 \sim \text{constant}. \quad (23)$$

as well as the simple estimate for the scattering time given above, Eq. (16). The scattering time is itself a function of time. It is natural in the present context to look for a power law dependence, $Q_s t_{\text{scat}} \sim (Q_s t)^a$. Then a simple analysis of the moments of the kinetic equation (12) reveals that the only sensible choice is $a = 1$ (provided one is not too close to equilibrium). We therefore set

$$t_{\text{scat}} \sim t. \quad (24)$$

With this assumption about the time dependence of the collision time, and imposing energy conservation, one easily determines the evolution of the two scales. We get

$$\Lambda_s \sim Q_s \left(\frac{t_0}{t} \right)^{\frac{3}{7}} \quad (25)$$

and

$$\Lambda \sim Q_s \left(\frac{t}{t_0} \right)^{\frac{1}{7}} \quad (26)$$

The number density of gluons $n_g \sim \Lambda^2 \Lambda_s$ decreases as $\sim (t_0/t)^{1/7}$, while the energy carried by the gluons, $\sim \Lambda_s \Lambda^3$ remains approximately constant. The Debye mass decreases slowly in time, $m \sim Q_s (t_0/t)^{1/7}$, so that indeed the energy carried by the condensate particles, with density $n_c \sim n_0 [1 - (t_0/t)^{1/7}]$, remains negligible,

$$\frac{\epsilon_c}{\epsilon_g} \sim \left(\frac{t_0}{t} \right)^{1/7}. \quad (27)$$

The thermalization time, determined from $\Lambda_s \sim \alpha_s \Lambda$, is

$$t_{\text{th}} \sim \frac{1}{Q_s} \left(\frac{1}{\alpha_s} \right)^{\frac{7}{4}}. \quad (28)$$

We notice that since the scale Λ is increasing in time, and therefore so is the entropy density $s \sim \Lambda^3$, there is entropy generation during the evolution, $s \sim \Lambda^3 \sim Q_s^3 (t/t_0)^{3/7}$. When $t = t_{\text{th}}$, $s \sim Q_s^3 / \alpha_s^{3/4}$, which is the equilibrium entropy $\sim T^3$. Note also that our expression (24) for the scattering time interpolates between $1/Q_s$, the scattering time in the initial plasma, and $\alpha_s^{-7/4} Q_s^{-1} \sim 1/(\alpha_s^2 T)$ in the equilibrated plasma. In fact, near thermalization, the scattering time may be given a familiar interpretation in terms of kinetic theory: $1/t_{\text{scat}} \sim \sigma n$, where $\sigma \sim \alpha_s^2 / \Lambda^2$, and $n \sim \Lambda^3$, which yields indeed $1/t_{\text{scat}} \sim \alpha_s^2 T$. Such an interpretation does not hold in the initial state where $1/t_{\text{scat}} \sim Q_s$, with no reference to the coupling constant.

Finally, the effect of quarks can be estimated. Quarks should have a phase space density of order 1 up to the scale Λ . Therefore the number density of quarks is

$$n_{\text{quarks}} \sim \Lambda^3 \quad (29)$$

Note that initially the quark number density is one order of α_s smaller as compared with that of the gluons $n_g \sim \Lambda_s \Lambda^2 / \alpha_s$. But by the thermalization time when $\Lambda_s \sim \alpha_s \Lambda$, they are of the same order and become equally important only at this time. Quarks may then, to first approximation, be ignored until the time of thermalization.

4 The effects of inelastic processes

Inelastic particle production or annihilation processes will modify the collision integral on the right hand side of the transport equation (12). Consider for example, the contribution of an $n \rightarrow m$ process to the collision term. The vertices contribute a factor α_s^{n+m-2} . There is a factor of $(\Lambda_s/\alpha_s)^{n+m-2}$ arising from the distribution functions (one factor for each distribution, except the one whose momentum one is following; besides, the products containing $n+m$ factors f cancel between the gain and loss terms). It follows that the coupling constant disappears explicitly. Furthermore, as shown for instance by Mueller et al. [12], there is an overall infrared singularity in the multiparticle production diagrams, which is cutoff by the Debye scale, and this yields a factor of $(1/m^2)^{n+m-4}$. Using $m^2 \sim \Lambda_s \Lambda$, we obtain overall a factor of $\Lambda_s^2/\Lambda^{n+m-4}$. This is balanced by a factor emerging from the remaining phase space integral. Since this integral is infrared finite, it is dominated by momenta of order Λ and is therefore proportional to a power of Λ . For dimensional reason, this is Λ^{n+m-5} , leaving an expression for the inelastic scattering which is parametrically identical to that for elastic scattering¹, namely $t_{\text{scat}} \sim \Lambda/\Lambda_s^2$.

Our procedure for determining Λ_s and Λ above used only $t_{\text{scat}} \sim t$ and energy conservation. Therefore, quite remarkably, including the effects of inelastic scattering does not change the scaling behaviour for Λ_s and Λ .

There are modifications of the treatment of the condensate however. As we have already mentioned, inelastic processes will inevitably lead to an equilibrium state without a condensate. The question then arises of whether such a condensate can exist as a transient state for a sufficient amount of time to influence the dynamics of the system. Of course, the answer to this question can only be obtained after a detailed numerical analysis of the solution of the transport equation. We can however offer the following lines of reasoning. In the small angle approximation to the transport equation for elastic processes, one finds, in case of overpopulation, that the gluon distribution develops very rapidly a $1/p$ behavior near $p = 0$, which eventually generates the delta-function singularity characteristic of the condensate. Unless the inelastic processes change this singular behavior of the distribution at small p , the elastic contribution to the collision integral provides therefore a source term for the condensate. There may be other source terms associated with higher order, multi-particle, processes. Inelastic scattering terms will also contribute a sink term. Since elastic and inelastic processes evolve with the same time scale, it is conceivable that a balance between the source and sink term can be achieved, and that a condensate is created and survives for most of the evolution till thermalization.

5 Effect of the longitudinal expansion

An important feature of the matter produced in ultra-relativistic heavy ion collisions is its strong longitudinal expansion. Assuming longitudinal boost invariance, and focussing on the central slice $z = 0$, one may capture the main effect of this expansion by adding to the left hand side of the kinetic equation a drift term of the form

$$\partial_\tau f - \frac{p_z}{t} \partial_{p_z} f = \left. \frac{df}{dt} \right|_{p_z t} = C[f], \quad (30)$$

¹We thank Guy Moore for pointing out a mistake in this analysis in an earlier version of this manuscript.

where the notation $\left. \frac{df}{dt} \right|_{p_z t}$ stands for a time derivative at constant $p_z t$. In the absence of the collision term, this equation admits free streaming solutions of the form $f(\mathbf{p}_\perp, p_z, t) = f(\mathbf{p}_\perp, p_z t/t_0)$. Thus, as an immediate effect of the longitudinal expansion, an isotropic initial distribution will flatten in the z -direction, on a time scale of order $t_0 \sim 1/Q_s$. This is an effect that will potentially delay complete three dimensional isotropization of the particles. We shall come back to this issue in the next section. Here, we shall make a simplifying assumption that the system may evolve under the combined effect of longitudinal expansion and collisions with a fixed anisotropy. We shall quantify shortly the degree of anisotropy.

By integrating over momentum the kinetic equation multiplied by the energy one obtains

$$\partial_t \epsilon + \frac{\epsilon + P_L}{t} = 0, \quad (31)$$

where ϵ is the energy density, $\epsilon = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} f_{\mathbf{p}}$, and P_L the longitudinal pressure, $P_L = \int \frac{d^3 p}{(2\pi)^3} \frac{p_z^2}{\omega_{\mathbf{p}}} f_{\mathbf{p}}$. One may analyze the effects of longitudinal expansion by parameterizing the longitudinal pressure in terms of energy density, namely, by assuming $P_L = \delta \epsilon$ where the multiplicative factor δ can be in the range $[0, 1/3]$ with $\delta = 0$ corresponding to the completely free streaming case and $\delta = 1/3$ corresponding to ideal hydrodynamic expansion after isotropization. Of course, the assumption that δ is independent of time is a strong assumption. We make it here in order to focus on the issue of how collisions redistribute momenta and thereby generate the shape of a thermal distribution for an expanding system. Note also that the nature of the local equilibrium changes somewhat as a function of δ . For instance, when $\delta = 0$ the local equilibrium is essentially two-dimensional, a situation that precludes the formation of a Bose condensate.

Within the present assumption the equation (31) becomes an equation for the evolution of the energy density

$$\epsilon_g(t) \sim \epsilon(t_0) \left(\frac{t_0}{t} \right)^{1+\delta}. \quad (32)$$

This equation, together with our previous estimate of the collision time (we can verify that the linear relation $t_{\text{scat}} \sim t$ remains unaffected by the expansion), yields the following estimates

$$\Lambda_s \sim Q_s \left(\frac{t_0}{t} \right)^{(4+\delta)/7}, \quad \Lambda \sim Q_s \left(\frac{t_0}{t} \right)^{(1+2\delta)/7}. \quad (33)$$

From these, one easily obtain the estimates of the gluon density, and of the Debye mass

$$n_g \sim \frac{Q_s^3}{\alpha_s} \left(\frac{t_0}{t} \right)^{(6+5\delta)/7}, \quad m^2 \sim Q_s^2 \left(\frac{t_0}{t} \right)^{(5+3\delta)/7}. \quad (34)$$

The thermalization time, obtained as before from the condition $\Lambda_s = \alpha_s \Lambda$, is given by

$$\left(\frac{t_{\text{th}}}{t_0} \right) \sim \left(\frac{1}{\alpha_s} \right)^{\frac{7}{3-\delta}}. \quad (35)$$

By comparing to the static case studied previously, we see that the expansion has the effect of delaying thermalization slightly. (Formally, one may recover the static case by setting $\delta = -1$, which corresponds to constant energy density.)

At this point, we may consider two scenarios. First, we assume particle number conservation. Then by integrating the transport equation over momenta, and noticing that the collision term does not contribute, we get

$$\partial_t n + \frac{n}{t} = 0, \quad n = n_0 \left(\frac{t_0}{t} \right) \sim \frac{Q_s^2}{\alpha_s} \frac{1}{t}. \quad (36)$$

Note that the effect of the expansion is to decrease the parameter $n\epsilon^{-3/4}$ that characterizes the overpopulation (ignoring thermalization processes beyond those responsible for maintaining isotropization):

$$n\epsilon^{-3/4} \sim \left(\frac{t_0}{t} \right)^{1/4} \left(\frac{t_0}{t} \right)^{-3\delta/4}. \quad (37)$$

Clearly, for $\delta = 1/3$, which corresponds to isotropic expansion, the decrease of the density and that of the energy density combine so as to leave the overpopulation parameter unchanged, while a fast decrease is achieved for the free streaming case $\delta = 0$. For moderate values of the anisotropy (more precisely for $\delta > 1/5$), a condensate can form, with density

$$n_c \sim \frac{Q_s^3}{\alpha_s} \left(\frac{t_0}{t} \right) \left[1 - \left(\frac{t_0}{t} \right)^{(-1+5\delta)/7} \right]. \quad (38)$$

As before, we can verify that the energy carried by the particles in the condensate is subleading:

$$\frac{\epsilon_c}{\epsilon_g} \sim \left(\frac{t_0}{t} \right)^{(5-11\delta)/14}, \quad (39)$$

which decreases with increasing t .

Alternatively, we may assume that particle number is not conserved. The analysis of this case follows that of the non expanding box, with the same uncertainty concerning the final role of inelastic scattering.

6 The asymmetry

At early times, a mechanism that attempts to restore isotropy between p_z and p_T are the Weibel instabilities [15–18,8,6].² These arise from purely imaginary screening masses in the presence of an asymmetric momentum distribution of gluons [19,20]. Generically, the time scale for the restoration of such symmetry is the inverse of the Debye mass, which we have argued to be $m \sim \sqrt{\Lambda\Lambda_s}$. Note that $1/m \sim 1/\sqrt{\Lambda\Lambda_s} \ll \Lambda/\Lambda_s^2 \equiv t_{\text{scat}}$ for times when $\Lambda \gg \Lambda_s$. This of course confirms that the time to restore isotropy is much less than the scattering time. If isotropy were maintained, one would require that $\delta = \frac{1}{3}$; the system would then evolve in time according to ideal hydrodynamics. However, as the system evolves in time, the Weibel instability, that operates on soft modes with momentum $p \leq m$ is likely not efficient enough at isotropizing hard modes with $p \sim \Lambda \gg m$. It is

²We will distinguish these instabilities from the initial “leading” instabilities that were responsible for isotropizing the system in the first place on a time scale of order $1/Q_s$ [14]. We however leave open the possibility that the subsequent dynamical evolution of the quantum system in ref. [14] may overlap at later times as well with the physics under discussion here.

possible that there may be some other mechanism similar to the Weibel instability that operates on higher momentum scale and generates isotropy there.

We may argue that scattering, while probably not sufficient to fully restore isotropy, may nevertheless maintain the system in a state of fixed anisotropy for a long time. For the sake of illustration, we can write down a simple kinetic equation that achieves this goal. To this aim, let us integrate the left hand side of the kinetic equation after multiplying it by either p_z^2 or p_\perp^2 . One gets

$$\int_{\mathbf{p}} p_z^2 \frac{df}{dt} \Big|_{p_z t} = \partial_t \langle p_z^2 \rangle + \frac{3}{t} \langle p_z^2 \rangle, \quad \int_{\mathbf{p}} p_\perp^2 \frac{df}{dt} \Big|_{p_z t} = \partial_t \langle p_\perp^2 \rangle + \frac{1}{t} \langle p_\perp^2 \rangle, \quad (40)$$

with $\langle p_z^2 \rangle \equiv \int_{\mathbf{p}} p_z^2 f$, and similarly for $\langle p_\perp^2 \rangle$. Defining the asymmetry p_A^2 in the momentum distribution by

$$p_A^2 \equiv p_z^2 - \frac{1}{2} p_\perp^2, \quad (41)$$

with $p_z^2 = (p^2 + 2p_A^2)/3$, and using the equations above for $\langle p_z^2 \rangle$ and $\langle p_\perp^2 \rangle$, we get

$$\int_{\mathbf{p}} p_A^2 \frac{df}{dt} \Big|_{p_z t} = \partial_t \langle p_A^2 \rangle + \frac{7}{3t} \langle p_A^2 \rangle + \frac{2}{3t} \langle p^2 \rangle. \quad (42)$$

It is plausible to assume that the collision term will contribute a “relaxation” force for the asymmetry. We therefore complete the equation as follows

$$\partial_t \langle p_A^2 \rangle + \frac{7}{3t} \langle p_A^2 \rangle + \frac{2}{3t} \langle p^2 \rangle = -\frac{\kappa}{t} \langle p_A^2 \rangle, \quad (43)$$

with κ a constant characterizing the strength of the collisions.

We will look for scaling solutions such that

$$\langle p^2 \rangle = \langle p^2 \rangle_0 \left(\frac{t_0}{t} \right)^\eta, \quad (44)$$

and further introduce a time-dependent dimensionless parametrization of the asymmetry $\langle p_A^2 \rangle$,

$$\langle p_A^2 \rangle = \zeta(t) \langle p^2 \rangle. \quad (45)$$

We find

$$t \partial_t \zeta + \left(\frac{7}{3} + \kappa - \eta \right) \zeta + \frac{2}{3} = 0. \quad (46)$$

As anticipated, this equation has a solution which relaxes towards a fixed value of ζ , namely

$$\zeta = -\frac{2/3}{7/3 + \kappa - \eta}. \quad (47)$$

One may relate (approximately) ζ to the parameter δ introduced earlier:

$$\delta \approx \langle p_z^2 \rangle / \langle p^2 \rangle = (1 + 2\zeta)/3. \quad (48)$$

One can then eliminate $\eta = (8 + 9\delta)/7$ and get ζ as a function of κ :

$$\zeta = \frac{16 + 21\kappa}{36} \left[1 - \sqrt{1 + \frac{1008}{(16 + 21\kappa)^2}} \right]. \quad (49)$$

Depending on the strength of the collisions, represented here by the parameter κ , various (negative, i.e. $\langle p_z^2 \rangle < \langle p_\perp^2 \rangle$) values of the anisotropy can be reached, from $\zeta \approx -1/2$ for small κ (in fact we must keep $\kappa > 1/7$ for δ to stay positive) to $\zeta = 0$ when $\kappa \rightarrow \infty$.

7 Summary

This paper argues that the Glasma formed in the early stages of heavy ion collisions is strongly interacting with itself up to parametrically late times when the system thermalizes. In particular, we show there are scaling solutions to the transport equations from which the coupling constant has disappeared. In addition, there may exist a transient component of the system, which is a Bose–Einstein condensate. If this scenario is realized, it may have a profound impact on the way in which we describe the properties of the Quark-Gluon Plasma in heavy ion collisions.

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